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An Anti-maximum Principle for Linear Elliptic Equations with an Indefinite Weight Function

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1. INTRODUCTION AND STATEMENT OF THE RESULT

In this note we prove an anti-maximum principle for the Dirichlet problem

$$\begin{aligned} \mathscr{L}u - \lambda mu &= h \qquad \text{in } \Omega, \\ u &= 0 \qquad \text{on } \partial\Omega, \end{aligned} \tag{1}$$

in the bounded domain $\Omega \subset \mathbb{R}^N$ $(N \geqslant 1)$ with smooth boundary $\partial\Omega$. By $\mathscr{L}$:

$$\mathscr{L}u = -\sum_{j,k=1}^{N} a_{jk} \frac{\partial^2 u}{\partial x_j \partial x_k} + \sum_{j=1}^{N} a_j \frac{\partial u}{\partial x_j} + a_0 u$$

we denote a strongly uniformly elliptic linear differential expression of second order with real-valued coefficient functions $a_{jk} = a_{kj}$, a_j, $a_0 \geqslant 0$ belonging to $C^\theta(\bar{\Omega})$ $(0 < \theta \leqslant 1)$; m and h are real-valued functions in $C(\bar{\Omega})$, and $\lambda \in \mathbb{R}$ a parameter. In particular, m may change sign in Ω. Let L_0 be the differential operator induced by $\mathscr{L}$ and the Dirichlet boundary conditions, with domain $D(L_0) = \{v \in C^{2+\theta}(\bar{\Omega}): v = 0 \text{ on } \partial\Omega\}$. Note that L_0 is closable in $C(\bar{\Omega})$ (it admits a closed extension in $L^p(\Omega)$, $1 < p < \infty$, having domain $W_0^{1,p}(\Omega) \cap W^{2,p}(\Omega)$). We set $L :=$ closure of L_0 in $C(\bar{\Omega})$. Then L is invertible. For the further study of L we introduce the real Banach spaces $E := C_0(\bar{\Omega}) := \{v \in C(\bar{\Omega}): v = 0 \text{ on } \partial\Omega\}$ and $X := C_0^1(\bar{\Omega}) := \{v \in C^1(\bar{\Omega}): v = 0 \text{ on } \partial\Omega\}$, with C-norm $\|\cdot\|_E$ and C^1-norm $\|\cdot\|_X$, respectively. By the L^p-theory for linear elliptic boundary value problems, L^{-1} maps $C(\bar{\Omega})$ compactly into $X \subset E$. Let finally $M: E \to E \subset C(\bar{\Omega})$ denote the multiplication operator by the function m.

We define u to be a solution of (1) provided

$$Lu - \lambda Mu = h. \tag{2}$$

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Suppose first that $m > 0$ on $\bar{\Omega}$, and let the spaces E and X be provided with the natural ordering given by the cones P_E and P_X of (pointwise) nonnegative functions. The standard notations of ordered Banach spaces are used: $w \geqslant 0$ if $w \in P$, $w > 0$ if $w \in P \backslash \{0\}$ ($P =$ positive cone). By the strong maximum principle, the compact operator $L^{-1}M$ maps positive functions in E into the interior of P_X (and hence into the quasi-interior of P_E [4, p. 241]). Thus $L^{-1}M: E \to E$ has positive spectral radius $\mu_1 := spr(L^{-1}M)$ [4, Th. 3.2, p. 270]. The Krein–Rutman theorem [4, Th. 3.2] guarantees that μ_1 is an eigenvalue of $L^{-1}M$; it is the only eigenvalue of $L^{-1}M$ whose associated eigenspace contains a positive function. Moreover the algebraic multiplicity of μ_1 equals 1. One calls $\lambda_1 := 1/\mu_1 > 0$ the principal eigenvalue of the equation

$$Lu = \lambda Mu; \tag{3}$$

by $u_1 > 0$ we denote the principal eigenfunction. It is proved in [3] that if $\hat{\lambda} \in \mathbb{C}$ is an eigenvalue of the problem obtained from (3) by complexification, then $\operatorname{Re} \hat{\lambda} \geqslant \lambda_1$.

If $\lambda < \lambda_1$, it is an immediate consequence of the positivity of the operator $L^{-1}M: E \to E$ that if u is a solution of Eq. (2) with $h \geqslant 0$, then $u \geqslant 0$ (for $\lambda \leqslant 0$ this holds by the maximum principle; for $0 < \lambda < \lambda_1$ we note that (2) is equivalent to

$$u = (I - \lambda L^{-1}M)^{-1}(L^{-1}h),$$

and since $L^{-1}h \geqslant 0$ and $spr(\lambda L^{-1}M) < 1$, the assertion follows by representing $(I - \lambda L^{-1}M)^{-1}$ as a Neumann series). For $m \equiv 1$ and $\lambda > \lambda_1$, Clément and Peletier [1] prove an interesting anti-maximum principle.

PROPOSITION 1 [1]. *Let $m \equiv 1$ and $h > 0$. Then there exists a number $\delta > 0$ (depending on h) such that $u < 0$ for the solution u of* (2) *with $\lambda \in (\lambda_1, \lambda_1 + \delta)$. More precisely, $u \leqslant c_\lambda u_1$, where $c_\lambda \to -\infty$ like $(\lambda_1 - \lambda)^{-1}$ as $\lambda \searrow \lambda_1$.*

We now admit functions $m \in C(\bar{\Omega})$ which may change sign. In [2] the hypotheses guaranteeing the existence of a positive principal eigenvalue λ_1 for (3) are drastically weakened. We state the main results obtained there.

PROPOSITION 2 [2]. *Suppose $m \in C(\bar{\Omega})$ is positive at some point in Ω. Then* (3) *admits a positive principal eigenvalue λ_1 characterized by being the unique positive eigenvalue having a positive eigenfunction u_1. Moreover $u_1 \in \operatorname{Int}(P_X)$, and λ_1 has the following properties*:

(i) *if* $\hat{\lambda} \in \mathbb{C}$ *is an eigenvalue of the problem obtained from* (3) *by complexification, with* $\operatorname{Re} \hat{\lambda} > 0$, *then* $\operatorname{Re} \hat{\lambda} \geqslant \lambda_1$;

(ii) $\mu_1 = 1/\lambda_1$ *is an eigenvalue of* $L^{-1}M: E \to E$ *with algebraic multiplicity* 1.

It is further shown that there is no eigenvalue $\hat{\lambda} \in \mathbb{C}$ with $\operatorname{Re} \hat{\lambda} = 0$.
Regarding Eq. (2), we have

PROPOSITION 3 [2]. *Suppose* $m \in (\bar{\Omega})$ *is positive somewhere in* Ω, *and* (2) *holds with* $h \geqslant 0$. *Then*

(a) *if* $0 \leqslant \lambda < \lambda_1$, *it follows that* $u \geqslant 0$;

(b) *if* $\lambda \geqslant \lambda_1$, $u > 0$ *is possible only in case* $\lambda = \lambda_1$ *and* $h = 0$.

The purpose of this note is to prove that Clément–Peletier's anti-maximum principle carries over to this much more general setting.

THEOREM. *Suppose* $m \in C(\bar{\Omega})$ *admits a positive value at some point in* Ω, *and let* $h > 0$ *be given in* $C(\bar{\Omega})$. *Then there exists a constant* $\delta = \delta(h) > 0$ *such that if* u *is the solution of* (2) *with* $\lambda \in (\lambda_1, \lambda_1 + \delta)$, *then* $u < 0$. *In fact,* $u \leqslant c_\lambda u_1$, *where* $c_\lambda \to -\infty$ *like* $(\lambda_1 - \lambda)^{-1}$ *as* $\lambda \searrow \lambda_1$.

Remark. We consider here only the Dirichlet problem since attention is restricted to Dirichlet boundary conditions in [2]. It is, however, possible to prove the results of [2] also for Neumann or regular oblique derivative boundary conditions.

2. PROOF OF THE THEOREM

We first note that Eq. (2) is equivalent to the equation

$$u - \lambda L^{-1}Mu = L^{-1}h \tag{4}$$

in E. The theorem is proved in a sequence of lemmata.

LEMMA 1. *The space* E *admits the topological direct decomposition*

$$E = \operatorname{span}[u_1] \oplus R(I - \lambda_1 L^{-1}M) \tag{5}$$

($u_1 > 0$ *is the principal eigenfunction associated with* λ_1).

Proof. Since $L^{-1}M: E \to E$ is compact, $I - \lambda_1 L^{-1}M$ is a Fredholm operator in E with index 0. Hence its range $R(I - \lambda_1 L^{-1}M)$ is closed, and $\operatorname{codim} R(I - \lambda_1 L^{-1}M) = \dim N(I - \lambda_1 L^{-1}M) = 1$. It thus suffices to show that $u_1 \notin R(I - \lambda_1 L^{-1}M)$.

Suppose, to the contrary, that $u_1 = (I - \lambda_1 L^{-1}M)w$ for some $w \in E$. Then $0 = (I - \lambda_1 L^{-1}M)^2 w$. Since $\mu_1 = 1/\lambda_1$ is eigenvalue of $L^{-1}M$ with algebraic multiplicity 1, it follows that $0 = (I - \lambda_1 L^{-1}M)w = u_1$, a contradiction. ∎

Let P denote the projection operator in E onto span$[u_1]$ parallel to $R(I - \lambda_1 L^{-1}M)$, and set $Q := I - P$. Writing

$$L^{-1}h = \alpha u_1 + g, \qquad u = \beta_\lambda u_1 + v_\lambda$$

($\alpha, \beta_\lambda \in \mathbb{R}$; $g, v_\lambda \in R(I - \lambda_1 L^{-1}M)$), Eq. (4):

$$(I - \lambda_1 L^{-1}M)u + (\lambda_1 - \lambda)\, L^{-1}Mu = L^{-1}h$$

is equivalent to the system

$$(\lambda_1 - \lambda)\, \lambda_1^{-1} \beta_\lambda u_1 + (\lambda_1 - \lambda)\, PL^{-1}Mv_\lambda = \alpha u_1 \tag{4a}$$

$$(I - \lambda_1 L^{-1}M)v_\lambda + (\lambda_1 - \lambda)\, QL^{-1}Mv_\lambda = g \tag{4b}$$

obtained according to the decomposition (5).

LEMMA 2. *There exist constants $\varepsilon > 0$ and $c > 0$ such that $|\lambda_1 - \lambda| < \varepsilon$ implies $\|v_\lambda\|_E \leqslant c$ for the solution $v_\lambda \in R(I - \lambda_1 L^{-1}M)$ of Eq.* (4b) *to the parameter value λ.*

Proof. Note that $I - \lambda_1 L^{-1}M$ is an isomorphism in the subspace $R := R(I - \lambda_1 L^{-1}M)$ of E. By the stability of bounded invertibility there exists $\varepsilon > 0$ such that the operator $(I - \lambda_1 L^{-1}M) + (\lambda_1 - \lambda)\, QL^{-1}M$ remains invertible in R provided $|\lambda_1 - \lambda| < \varepsilon$. The assertion thus follows from Eq. (4b). ∎

We need an estimate slightly stronger than that given by Lemma 2. Observe that if $v_\lambda \in R(I - \lambda_1 L^{-1}M)$ is solution of Eq. (4b), then $v_\lambda \in X$. In fact, $g = L^{-1}h - \alpha u_1 \in X$; further $L^{-1}Mv_\lambda \in X$ and $QL^{-1}Mv_\lambda = (I - P)L^{-1}Mv_\lambda \in X$ since $PL^{-1}Mv_\lambda = d_\lambda u_1 \in X$ ($d_\lambda \in \mathbb{R}$).

LEMMA 3. *The assertion of Lemma 2 holds with the E-norm replaced by the X-norm.*

Proof. By Lemma 2 we know that $\|v_\lambda\|_E \leqslant c$ for the solution v_λ of (4b) provided $|\lambda_1 - \lambda| < \varepsilon$. Hence $\|L^{-1}Mv_\lambda\|_X \leqslant c_1$ and $\|QL^{-1}Mv_\lambda\|_X \leqslant c_1 + |d_\lambda|\, \|u_1\|_X \leqslant c_2$, since $|d_\lambda| \leqslant d$ by the boundedness of $\{PL^{-1}Mv_\lambda\}$ in E. The result follows again from Eq. (4b). ∎

Since $u_1 \in \mathrm{Int}(P_X)$ we infer the existence of a constant $\gamma \in \mathbb{R}$ such that $v_\lambda \leqslant \gamma u_1$ for the solution v_λ of (4b) to values λ with $|\lambda_1 - \lambda| < \varepsilon$. Equation (4a) implies

$$\beta_\lambda = \alpha \lambda_1 (\lambda_1 - \lambda)^{-1} - \lambda_1 d_\lambda \qquad (\lambda \neq \lambda_1);$$

thus

$$u \leqslant c_\lambda u_1 \; (0 < |\lambda_1 - \lambda| < \varepsilon), \tag{6}$$

where $c_\lambda = \beta_\lambda + \gamma \leqslant \alpha\lambda_1(\lambda_1 - \lambda)^{-1} + \lambda_1 d + \gamma$.

LEMMA 4. *In the decomposition* $L^{-1}h = \alpha u_1 + g$ $(g \in R(I - \lambda_1 L^{-1}M))$, $h > 0$ *implies* $\alpha > 0$.

Proof. In the present generality we are not able to prove Lemma 4 directly. We show that the cases $\alpha < 0$ and $\alpha = 0$ are impossible.

(i) Assume $\alpha < 0$. Then (6) implies that $u \leqslant c_\lambda u_1$ with $c_\lambda \to -\infty$ as $\lambda \nearrow \lambda_1$, contradicting Proposition 3(a).

(ii) Suppose $\alpha = 0$, i.e., $L^{-1}h = w - \lambda_1 L^{-1}Mw$ for some $w \in E$. Without loss of generality we may assume $|m| < 1$ on $\bar{\Omega}$. Then

$$h = Lw - \lambda_1 Mw = (L + \lambda_1)w - \lambda_1(M + 1)w,$$

which gives

$$(L + \lambda_1)^{-1}h = w - \lambda_1 K_{\lambda_1} w, \tag{7}$$

where $K_{\lambda_1} := (L + \lambda_1)^{-1}(M + 1)\colon E \to E$ is now a compact *positive* operator. Since $\mu_1 u_1 = K_{\lambda_1} u_1$ $(\mu_1 = 1/\lambda_1, u_1 > 0)$, we conclude by the Krein–Rutman theorem that μ_1 is also an eigenvalue of $K^*_{\lambda_1}\colon E^* \to E^*$ (the Banach space adjoint operator), with positive eigenfunction u_1^*. From (7) we infer that

$$(L + \lambda_1)^{-1}h \in R(I - \lambda_1 K_{\lambda_1}) = N(I - \lambda_1 K^*_{\lambda_1})^\perp = (\operatorname{span}[u_1^*])^\perp;$$

consequently

$$\langle u_1^*, (L + \lambda_1)^{-1}h \rangle = 0.$$

On the other hand, $(L + \lambda_1)^{-1}h$ is a quasi-interior point of P_E and hence (since $u_1^* > 0$)

$$\langle u_1^*, (L + \lambda_1)^{-1}h \rangle > 0.$$

This contradiction shows that $\alpha \neq 0$. ∎

The assertion of the Theorem now follows immediately from (6) with $\alpha > 0$.

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